Using Model Checking with Symbolic Execution to Verify Parallel Numerical Programs

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Parallel numerical programs

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The Problem:

Parallel numerical programs are very difficult to get right.

Matrix multiplication: sequential version

```
double A[N][L], B[L][M], C[N][M];
            ⋮
int i,j,k;
for (i=0; i<N; i++)
  for (j=0; j<M; j++) {
    C[i][j] = 0.0;
    for (k=0; k<L; k++)
      C[i][j] += A[i][k]*B[k][j];
  }
```

Matrix multiplication: parallel version (master-slave)

```
int rank,nprocs,i,j,numsent,sender,row,anstype;
double buffer[L], ans[M];
MPI_Status status;
MPI_Comm_size(MPI_COMM_WORLD, &nprocs);
MPI_Comm_rank(MPI_COMM_WORLD, &rank);
if (rank==0) { /* I am the master */
  numsent=0;
  for (i=0; i<nprocs-1; i++) {
    for (j=0; j<L; j++)
      buffer[j] = A[i][j];
    MPI_Send(buffer, L, MPI_DOUBLE, i+1,
      i+1, MPI_COMM_WORLD);
    numsent++;
  }
  for (i=0; i<N; i++) {
    MPI_Recv(ans, M, MPI_DOUBLE, MPI_ANY_SOURCE,
      MPI_ANY_TAG, MPI_COMM_WORLD, &status);
    sender = status.MPI_SOURCE;
    anstype = status.MPI_TAG-1;
    for (j=0; j<M; j++)
      C[anstype][j] = ans[j];
    if (numsent<N) {
      for (j=0; j<L; j++)
        buffer[j] = A[numsent][j];
      MPI_Send(buffer, L, MPI_DOUBLE, sender,
        numsent+1, MPI_COMM_WORLD);
      numsent++;
    }
    else MPI_Send(buffer, 1, MPI_DOUBLE, sender,
      0, MPI_COMM_WORLD);
  }
} else { /* I am a slave */
  while (1) {
    MPI_Recv(buffer, L, MPI_DOUBLE, 0,
      MPI_ANY_TAG, MPI_COMM_WORLD, &status);
    if (status.MPI_TAG==0) break;
    row = status.MPI_TAG-1;
    for (i=0; i<M; i++) {
      ans[i] = 0.0;
      for (j=0; j<L; j++)
        ans[i] += buffer[j]*B[j][i];
    }
    MPI_Send(ans, M, MPI_DOUBLE, 0,
      row+1, MPI_COMM_WORLD);
  }
}
```

adapted from

Using MPI

by

William Gropp
Ewing Lusk
Anthony Skjellum

Why parallel numerical programs are difficult to get right

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- usual reasons concurrent programming is difficult
  - parallelism adds complexity
  - nondeterminism
  - deadlocks
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- additional sources of nondeterminism from MPI
- problem of test oracles
  - in scientific computation, often don't know correct result for a given test input, so can't tell if the observed result is correct
  - analytical solutions rarely exist
  - sequential program may only work on small test cases
  - floating-point arithmetic differs from real arithmetic

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3. rewriting code in the hope that the problem will disappear
   - insertion of barriers

Our contribution

A method to verify parallel numerical programs.

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- uses symbolic execution to model floating-point computation
- uses model checking
  - requires translation of program into input language of model checker
  - verifies equivalence over all executions
  - produces a trace if program is incorrect

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- one double-precision floating-point variable has 2^{64} possible states
- abstraction?

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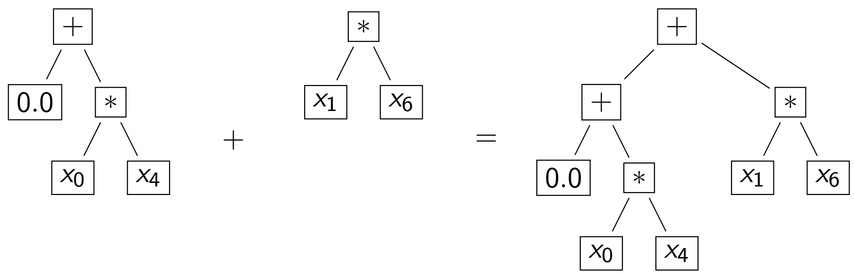
Input: symbolic constants $x_0, x_1, \ldots$
Output: symbolic expressions in the x_i



$$0.0 + (x_0 x_4) \quad + \quad x_1 x_6 \quad = \quad (0.0 + (x_0 x_4)) + x_1 x_6$$

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- in the model...
  - replace all floating-point values with ID numbers
  - replace all floating-point operations with symbolic operations
    - to evaluate $x + y$:
    - is $x + y$ already in the table?
    - if *yes*, return its ID number
    - if *no*, create new table entry and return new ID number

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$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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9	$(\mathtt{X}, 7)$	x_7
10	$(\mathtt{*}, 2, 6)$	$x_0 x_4$
11	$(\mathtt{+}, 0, 10)$	$0.0 + x_0 x_4$

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5	(X, 3)	x_3
6	(X, 4)	x_4
7	(X, 5)	x_5
8	(X, 6)	x_6
9	(X, 7)	x_7
10	(*, 2, 6)	$x_0 x_4$
11	(+, 0, 10)	$0.0 + x_0 x_4$
12	(*, 3, 8)	$x_1 x_6$

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15	(+, 0, 14)	$0.0 + x_0 x_5$
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  - **if** $(x_0 \neq 0)$ $\{\ldots\}$ **else** $\{\ldots\}$

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- the programs may contain branches on expressions that involve the symbolic variables
  - **if** $(x_0 \neq 0)$ $\{\ldots\}$ **else** $\{\ldots\}$
- only want to compare the result of an execution path in the parallel program to the result of a *corresponding* path in the sequential program

Path conditions and domains

- enumerate all paths through the sequential program
  - keeping track of the path condition for each path

$$\mathbf{y} = \begin{cases} f_1(\mathbf{x}) & \text{if } p_1(\mathbf{x}) \\ f_2(\mathbf{x}) & \text{if } p_2(\mathbf{x}) \\ \vdots & \vdots \\ f_n(\mathbf{x}) & \text{if } p_n(\mathbf{x}) \end{cases}$$

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- each p_i determines a path domain $D_i = \{\mathbf{x} \mid p_i(\mathbf{x})\}$
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Solution to path correspondence problem:

1. discover path conditions/domains automatically
2. for each domain D_i: compare symbolic results of sequential and parallel programs for all inputs in D_i

Modeling conditional statements

To model the statement **if** $(x_0 \neq 0)\ \{\ldots\}$ **else** $\{\ldots\}$

$p \leftarrow$ **true**; /* path condition */
$\vdots$
$b \leftarrow \mu(p, x_0 \neq 0)$;
if $(b = -1)$ {
  if $(choose())$ {
    $b \leftarrow 1$; $p \leftarrow p \wedge (x_0 \neq 0)$;
  } **else** {
    $b \leftarrow 0$; $p \leftarrow p \wedge (x_0 = 0)$;
  }
}
if $(b = 1)$ { ... } **else** { ... }

$$\mu(p, q) = \begin{cases} 1 & \text{if } p \Rightarrow q \\ 0 & \text{if } p \Rightarrow \neg q \\ -1 & \text{if don't know} \end{cases}$$

for boolean-valued symbolic expressions p, q

The method

1. construct symbolic model M_{seq} of sequential program
   - input: $\mathbf{x}$, output: $\mathbf{y}$, path condition: p
2. construct symbolic model M_{par} of parallel program
   - input: $\mathbf{x}$, output: $\mathbf{y}'$, path condition: p
   - using same symbolic table
3. create composite model M:
   - $p \leftarrow \textbf{true}$; M_{seq}; M_{par}; $\texttt{assert(}\mathbf{y} = \mathbf{y}'\texttt{);}$
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The model checker returns either

- Yes: the property holds, or
- No + counterexample:
  - a trace through M_{seq}
  - a trace through M_{par}
  - the values of p, $\mathbf{y}$, and $\mathbf{y}'$

Numerical Issues

different symbolic expressions are equivalent over real numbers

- example: $((x_3 + x_1) + x_2) + x_0$ and $((x_0 + x_1) + x_2) + x_3$

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  3. Real
     - all of IEEE
     - $(x + y) + z = x + (y + z)$
     - $(xy)z = x(yz)$
     - ...

Preliminary experimental results

- implemented as extension to SPIN
- wrote our own simple symbolic algebra library and light-weight theorem-prover
- found bug in one example (Jacobi iteration)

	matrix multiplication	Gaussian elimination	Jacobi iteration	Monte Carlo
processes	6	6	17	9
path domains	1	13327	4	4
symbolic expressions	2202	247656	8239	1232
input vector size	200	36	1333	99
output vector size	100	36	36	1
states (10^3)	4443	16114	6295	3112
memory (MB)	217	801	362	279
time (s)	506	3224	9846	738

Related Work

1. Ball and Rajamani. *Automatically validating temporal safety properties of interfaces*. SPIN 2001.
2. Khurshid, Păsăreanu, and Visser. *Generalized symbolic execution for model checking and testing*. TACAS 2003.
3. Păsăreanu and Visser. *Verification of Java programs using symbolic execution and invariant generation*. SPIN 2004.
4. Elmas, Tasiran, and Qadeer. *VYRD: verifying concurrent programs by runtime refinement-violation detection*. PLDI 2005.

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- models are constructed by hand (for now)
- state explosion
  - often requires small bounds on configuration
- possibility of spurious error report
  - precision depends upon close correspondence between numerical operations in sequential and parallel programs

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ & \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

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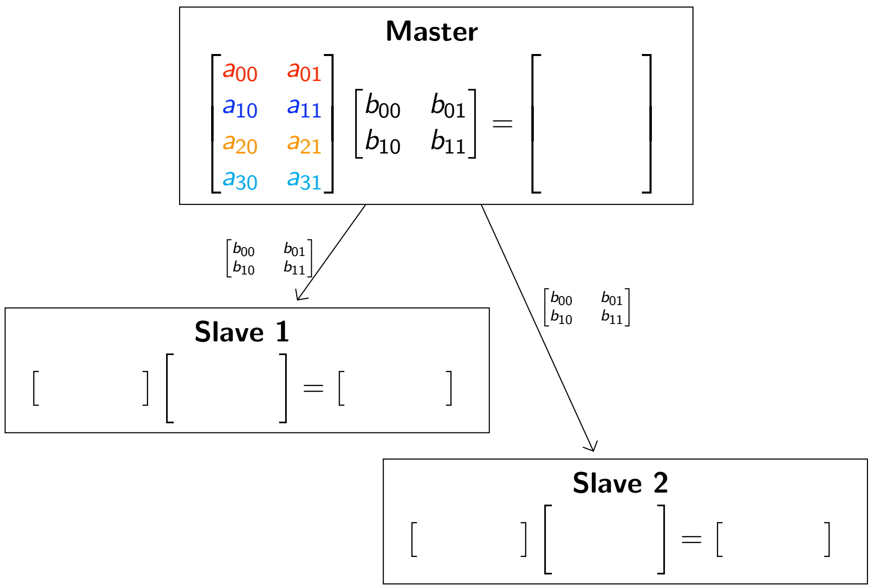

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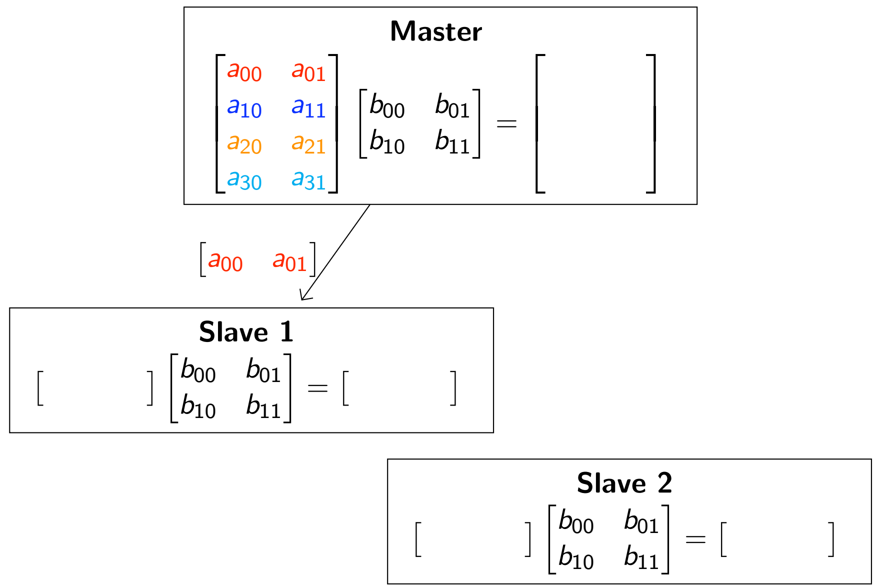

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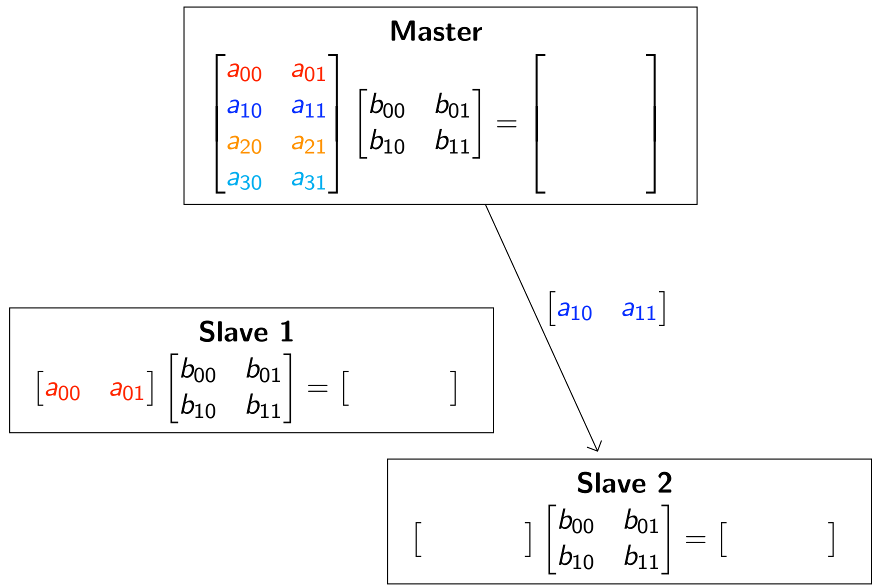

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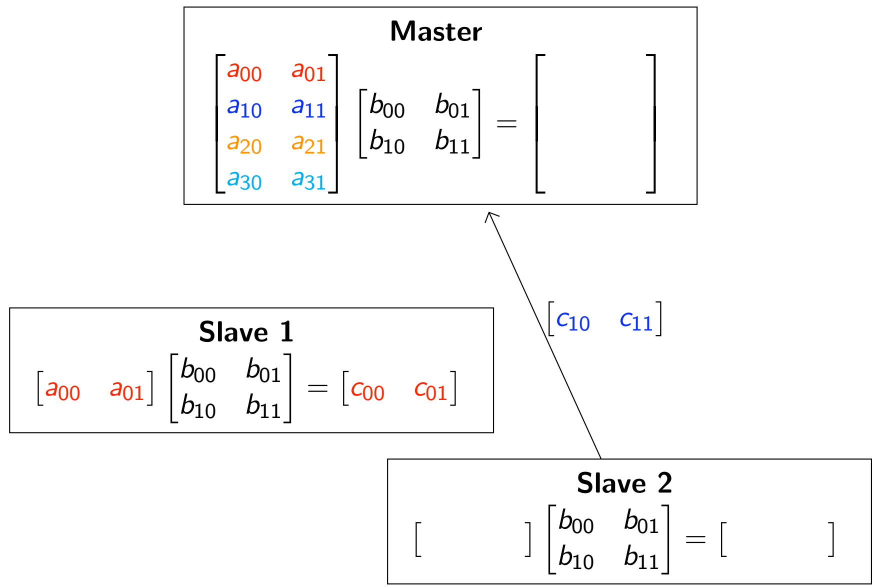

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Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{10} & c_{11} \end{bmatrix}$$

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} a_{00} & a_{01} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} a_{00} & a_{01} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} c_{01} & c_{11} \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

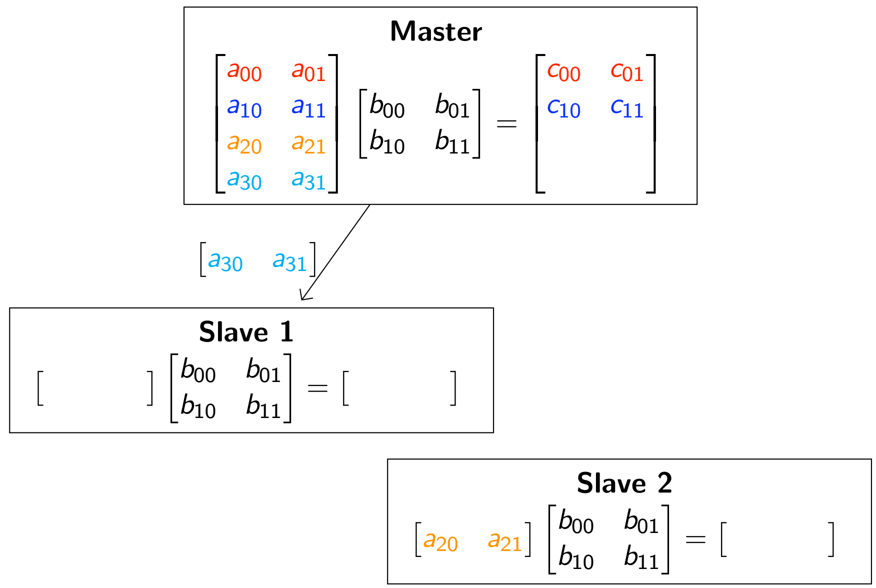
Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication



Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

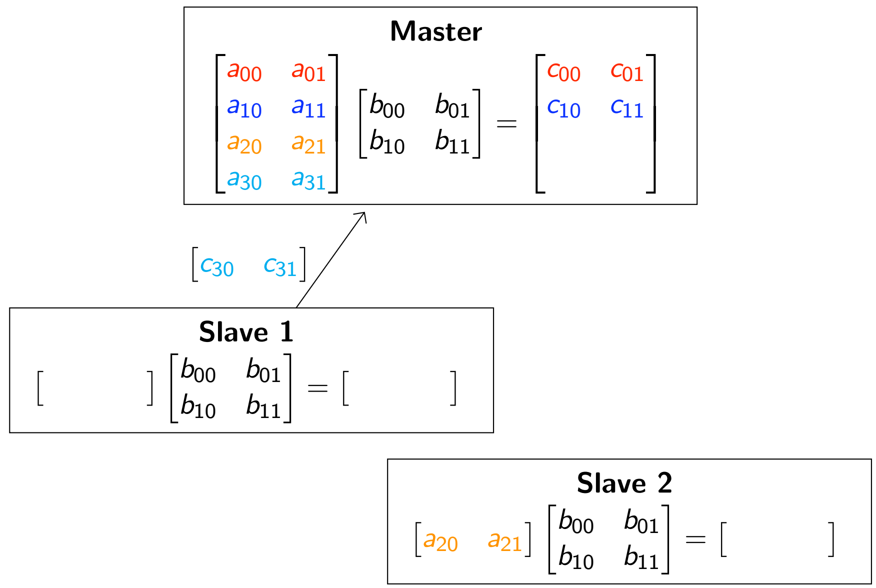
Slave 1

$$\begin{bmatrix} a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{30} & c_{31} \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication



Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ & \\ c_{30} & c_{31} \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ & \\ c_{30} & c_{31} \end{bmatrix}$$

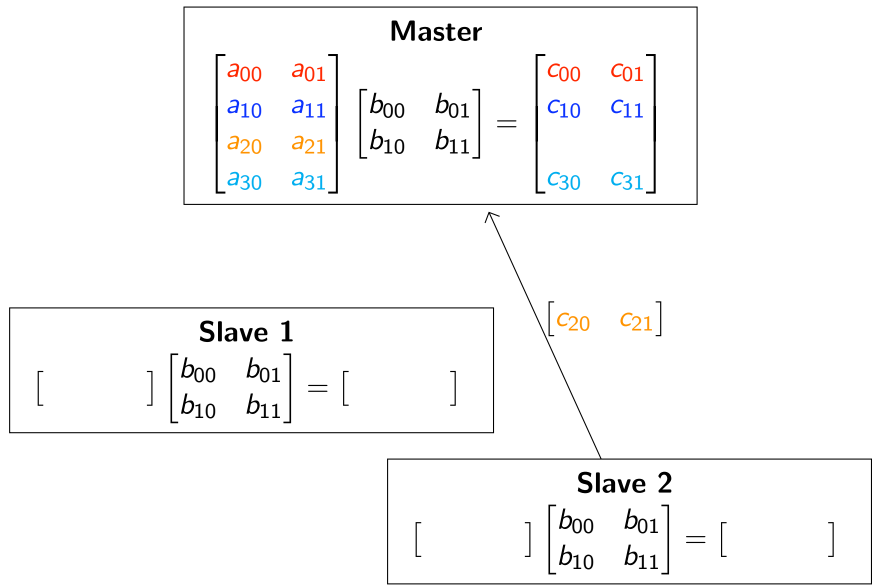
Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{20} & c_{21} \end{bmatrix}$$

Master-slave matrix multiplication



Master-slave matrix multiplication

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ c_{20} & c_{21} \\ c_{30} & c_{31} \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

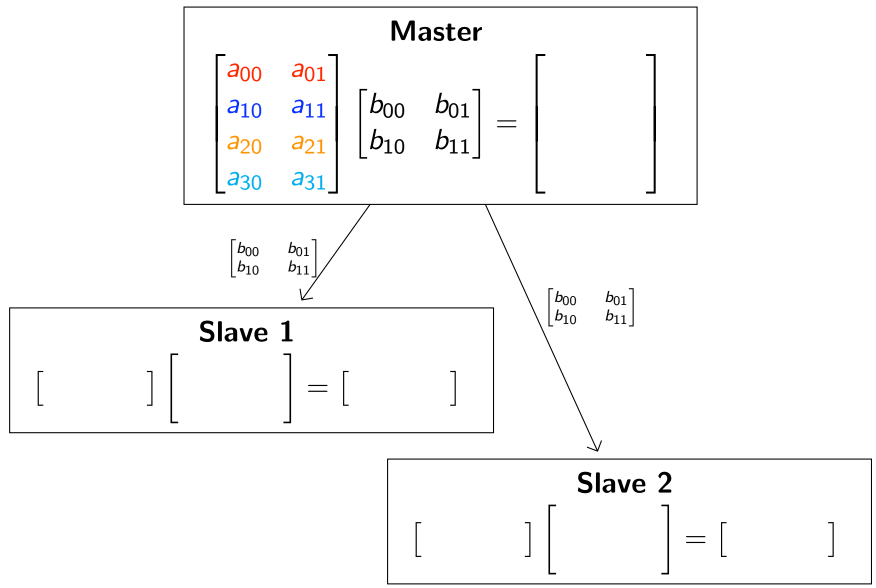
Slave 1

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

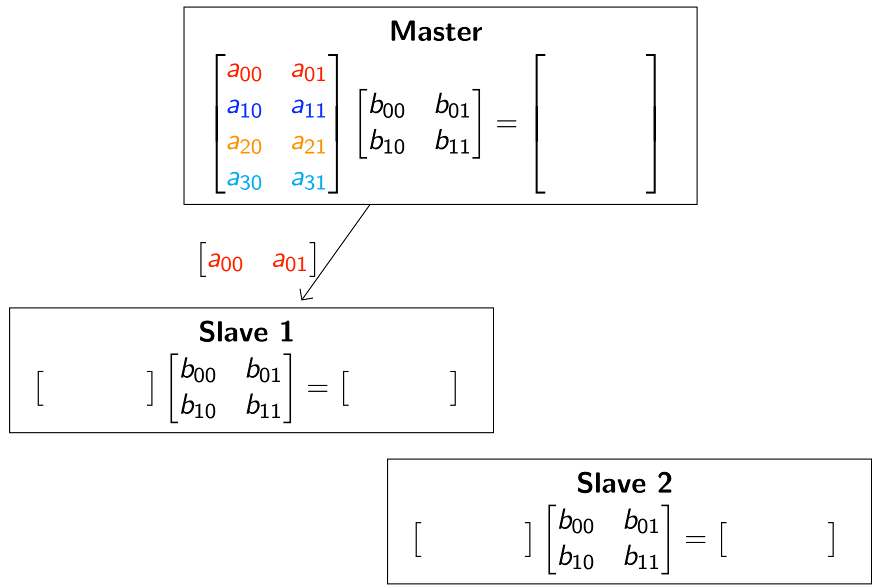
Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}$$

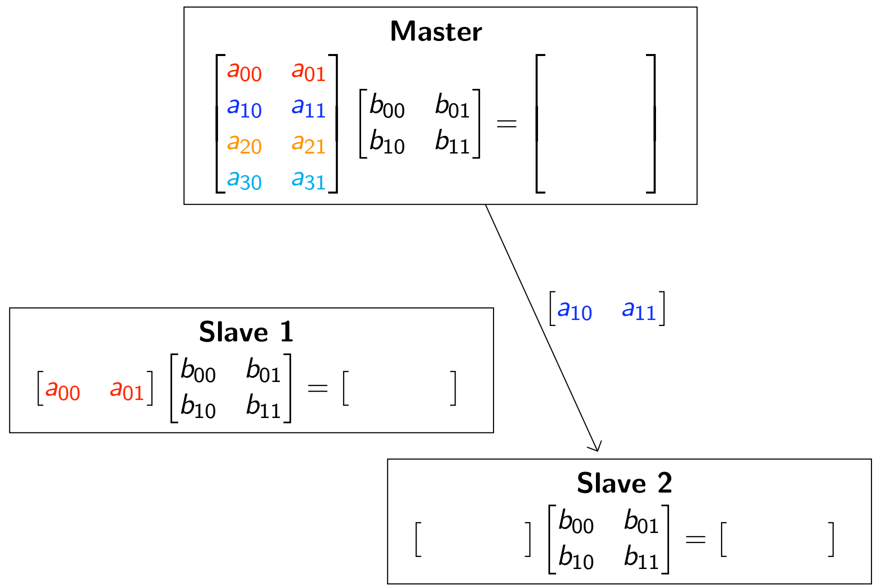
Slave 1

$$\begin{bmatrix} a_{00} & a_{01} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} a_{00} & a_{01} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} a_{00} & a_{01} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{10} & c_{11} \end{bmatrix}$$

Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} c_{01} & c_{11} \end{bmatrix}$$

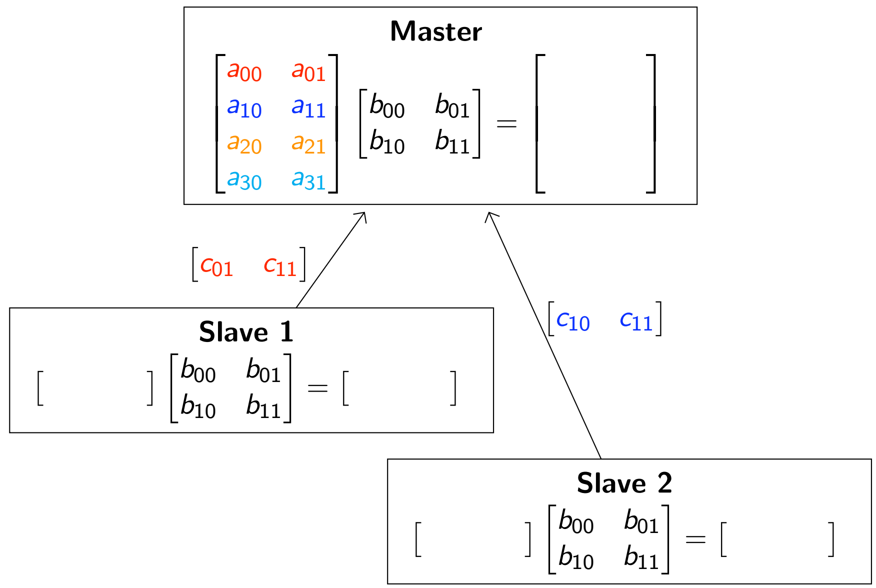
Slave 1

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}$$

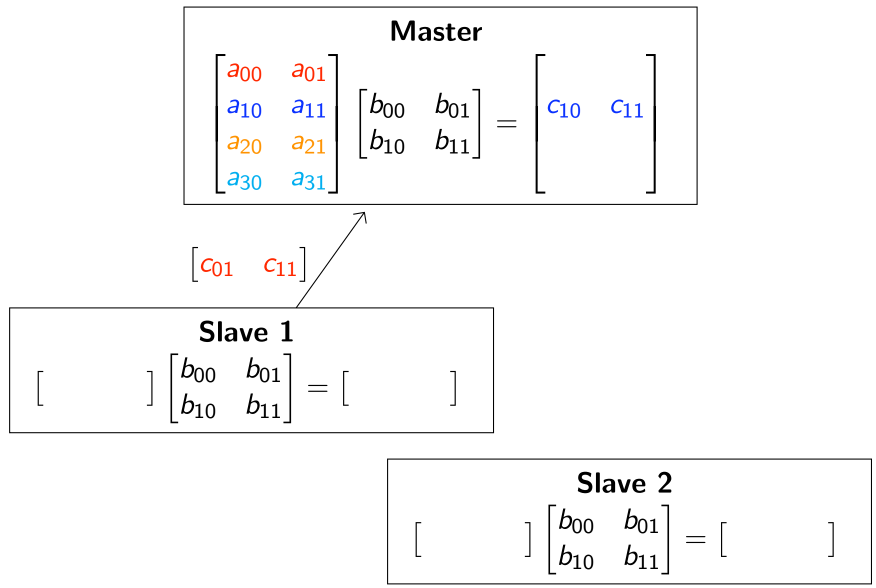
Slave 2

$$\begin{bmatrix} a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{10} & c_{11} \end{bmatrix}$$

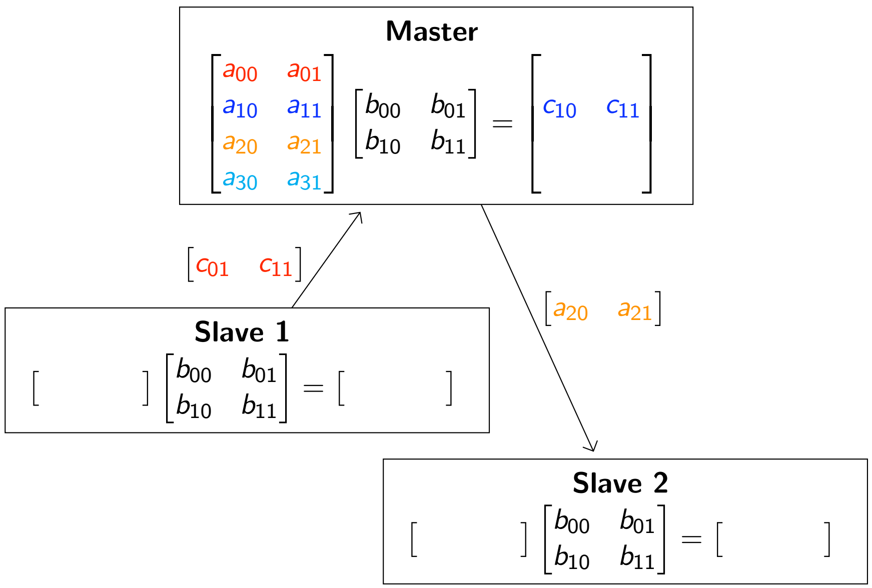
Master-slave matrix multiplication: execution 2



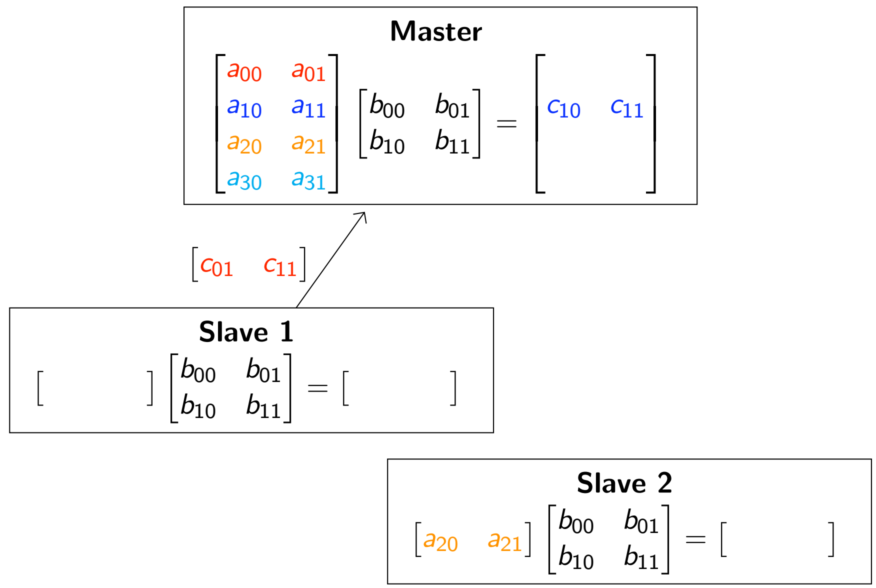
Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} & \\ c_{10} & c_{11} \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} c_{01} & c_{11} \end{bmatrix}$$

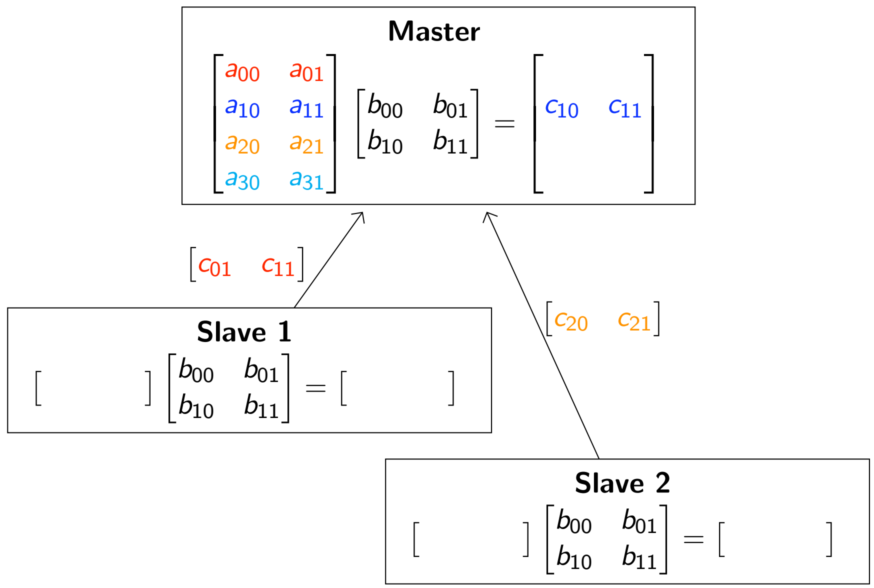
Slave 1

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

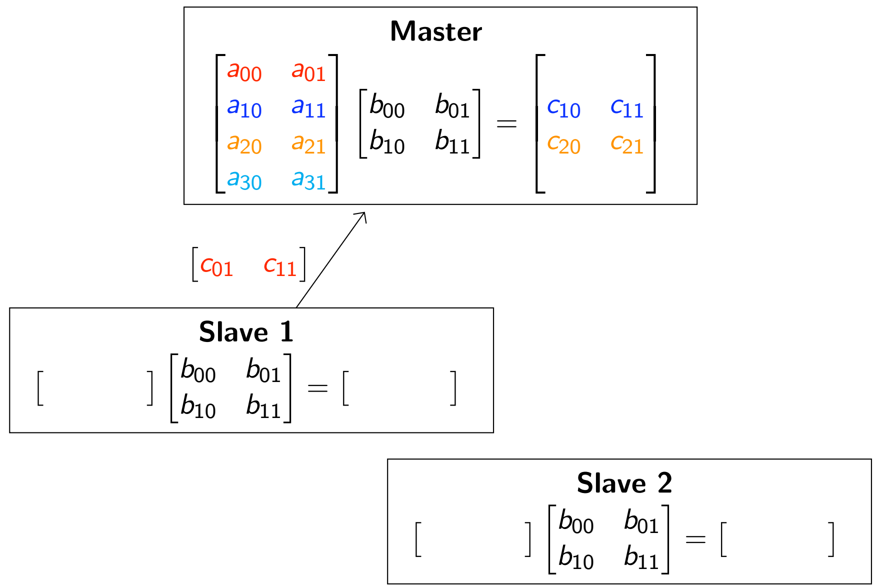
Slave 2

$$\begin{bmatrix} a_{20} & a_{21} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{20} & c_{21} \end{bmatrix}$$

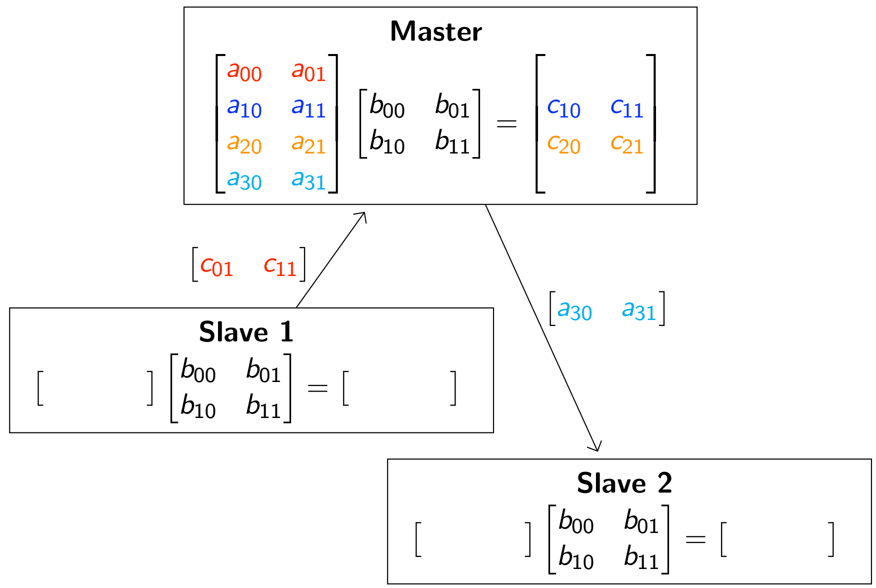
Master-slave matrix multiplication: execution 2



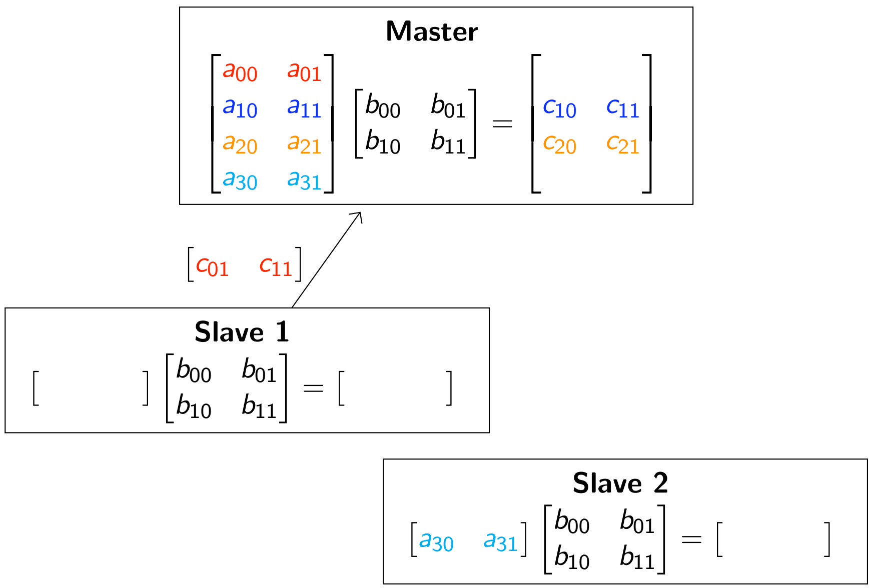
Master-slave matrix multiplication: execution 2



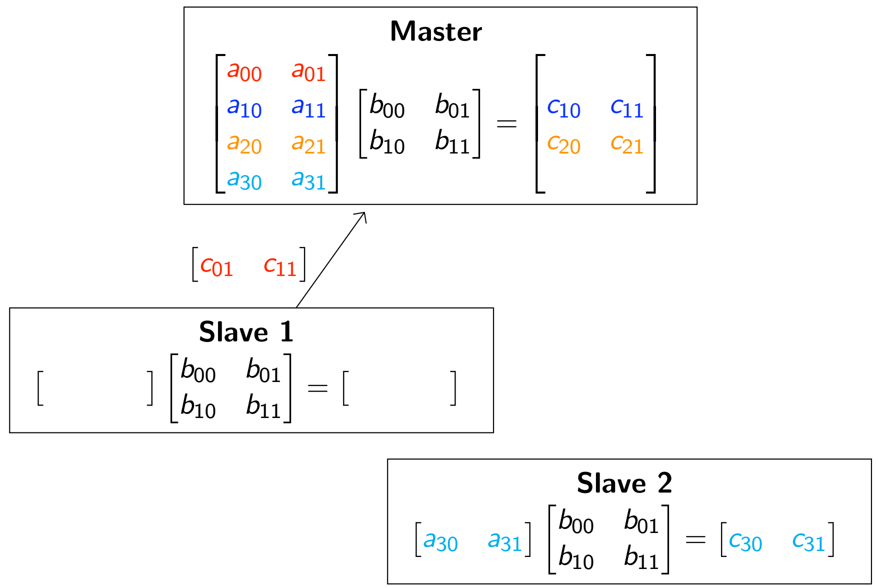
Master-slave matrix multiplication: execution 2



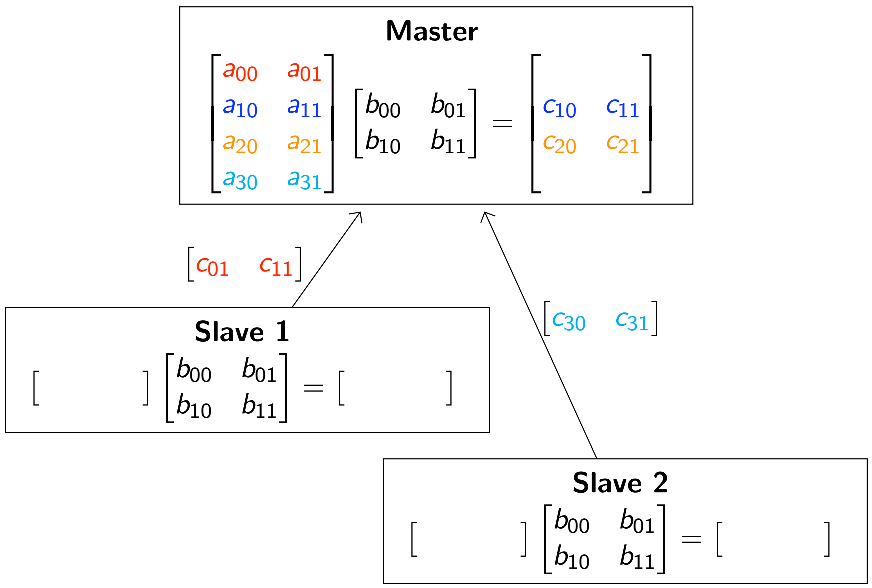
Master-slave matrix multiplication: execution 2



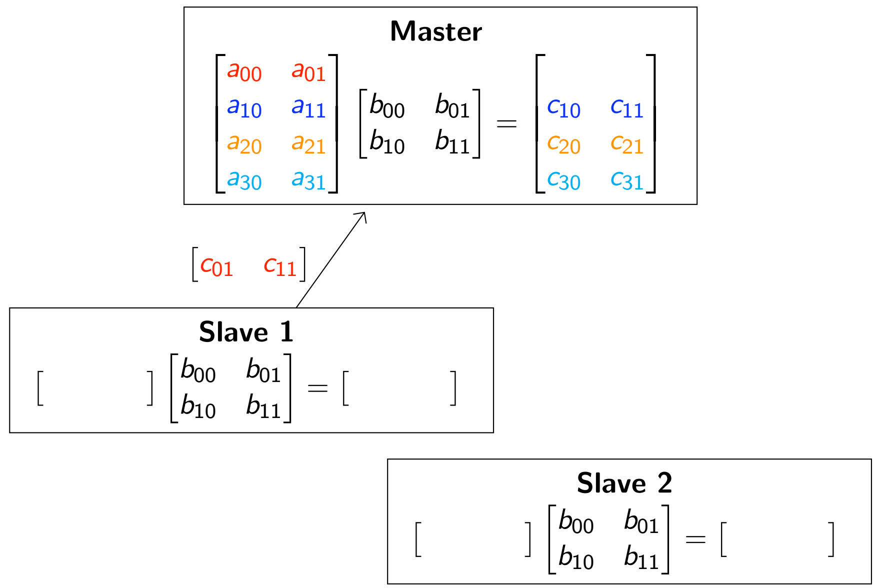
Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2



Master-slave matrix multiplication: execution 2

Master

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \\ c_{20} & c_{21} \\ c_{30} & c_{31} \end{bmatrix}$$

Slave 1

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Slave 2

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Gaussian elimination

Step 1 Locate the leftmost column of A that does not consist entirely of zeros, if one exists. The top nonzero entry of this column is the pivot.

Step 2 Interchange the top row with the pivot row, if necessary, so that the entry at the top of the column found in Step 1 is nonzero.

Step 3 Divide the top row by pivot in order to introduce a leading 1.

Step 4 Add suitable multiples of the top row to all other rows so that all entries above and below the leading 1 become zero.

Repeat.

Gaussian elimination

transforms a matrix to its reduced row-echelon form:

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \quad \longrightarrow \quad \mathbf{y} = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}$$

Gaussian elimination

transforms a matrix to its reduced row-echelon form:

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \quad \longrightarrow \quad \mathbf{y} = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix}$$

$$\mathbf{y} = \begin{cases} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \land x_2 = 0 \land x_1 = 0 \land x_3 = 0 \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \land x_2 = 0 \land x_1 = 0 \land x_3 \neq 0 \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \land x_2 = 0 \land x_1 \neq 0 \\ \left(\begin{smallmatrix} 1 & x_3/x_2 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 = 0 \land x_2 \neq 0 \land x_1 = 0 \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } x_0 = 0 \land x_2 \neq 0 \land x_1 \neq 0 \\ \left(\begin{smallmatrix} 1 & x_1/x_0 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } x_0 \neq 0 \land x_3 - x_2(x_1/x_0) = 0 \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } x_0 \neq 0 \land x_3 - x_2(x_1/x_0) \neq 0 \end{cases}$$